Free random Lévy matrices

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Using the theory of free random variables and the Coulomb gas analogy, we construct stable random matrix ensembles that are random matrix generalizations of the classical one-dimensional stable Lévy distributions. We show that the resolvents for the corresponding matrices obey transcendental equations in the large size limit. We solve these equations in a number of cases, and show that the eigenvalue distributions exhibit Lévy tails. For the analytically known Lévy measures we explicitly construct the density of states using the method of orthogonal polynomials. We show that the Lévy tail distributions are characterized by a different novel form of microscopic universality.

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There is a wide and growing interest in stochastic processes with long tails in relation to self-similar phenomena. Long tails cause the two-dimensional random walk to be attracted to Lévy stable fixed points with ubiquitous physical properties such as intermittent behavior and anomalous diffusion. Physical examples include charge carrier transport in amorphous semiconductors, vortex motion in high temperature superconductors, moving interfaces in porous media, spin glasses, and anomalous heat flow in heavy ion collisions [1–4]. Lévy distributions have also found applications in biophysics, health physics, and finances [5–7]. Their importance stems from their stability under convolution, i.e., the sum of two Lévy distributed random variables follows also a Lévy distribution.

In the realm of complex and/or disordered systems, the theory of random matrices plays an important role in differentiating noise from information. Also, it allows for a generic analysis of complex phenomena in the chaotic regime using random matrix universality. So far, there does not seem to be a developed theory of random Lévy matrices, with the exception of [8] that is not based on free random variables (FRV). The reasons are twofold. Lévy distributions are usually defined by their characteristic functions, while their probability density functions are very complicated and can usually be expressed only indirectly through integrals. Second, Lévy distributions do not have finite second and higher moments, making standard techniques of random matrix theory usually of little use.

The aim of this paper is to provide an appropriate generalization of the classical stable one-dimensional Lévy distributions to the random matrix setting and show a different form of universal scaling in the tails.

The fundamental problem in random matrix theory is to find the distribution of eigenvalues λ_i in the large N (size of M) limit, i.e.,

$$\rho(\lambda)=\frac{1}{N}\left\langle\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\rangle=\frac{1}{N}\langle\mathrm{Tr}\,\delta(\lambda-M)\rangle. \tag{1}$$

In this paper we are interested in stable random matrix ensembles, i.e., ensembles where the averaging is carried using some pertinent measure of the form

$$e^{-N\mathrm{Tr}\,V(M)}dM, \tag{2}$$

Moreover, we require that the eigenvalue distribution of the sum $M=M_1+M_2$, where both M_1 and M_2 have the measure of the form (2), is the same (up to a shift and/or rescaling) as for each individual matrix M_1 and M_2. We emphasize that we will require this property to be necessarily true only in the limit $N\to\infty$. If the second moment exists, then the ensemble is Gaussian. In the opposite case there exist, however, alternate ensembles that are the matrix analogs of the classical one-dimensional Lévy distributions. The potential $V(M)$ need not be analytic in M (see below). In general, it is convenient to introduce the Green's function

$$G(z)=\frac{1}{N}\left\langle\mathrm{Tr}\frac{1}{z-M}\right\rangle. \tag{3}$$

The eigenvalue distribution follows from the discontinuity of $G(z)$ along the real axis, i.e., $\rho(\lambda)=-\mathrm{Im}\,G(\lambda+i0)/\pi$.

The notion of addition laws of the type $M=M_1+M_2$ has been analyzed in the context of the theory of FRV, a generalization by Voiculescu [9] of classical probability theory to a noncommutative setting. The abstract concepts of operator algebras and free random variables can have an explicit realization in terms of large random matrices. Hence, FRV techniques provide a novel and powerful way of analyzing the spectra of random matrices [10,11]. In this framework the determination of stable *eigenvalue distributions* can be performed algebraically in a very general setting. This has been done by Bercovici and Voiculescu [12]. However, the drawback of such an abstract approach is that the explicit random matrix ensembles corresponding to these distributions are unknown. The aim of this paper is to fill this gap, as the knowledge of explicit stable random matrix ensembles with power-law tails might be interesting for the various applications we have cited.

The remarkable achievement by Bercovici and Voiculescu [12] is an explicit derivation of all R transforms {defined by the equation $R[G(z)]=z-1/G(z)$ where $G(z)=\langle 1/(z-\lambda)\rangle$} for all free stable distributions, without recourse to a matrix realization. Indeed, Bercovici and Voiculescu have found that $R(z)$ can have the trivial form $R(z)=a$ or

$$R(z)=a+bz^{\alpha-1}, \tag{4}$$

where $0<\alpha<2$ and $\alpha\neq 1$, a is a real shift parameter, and b is a parameter that can be related to the slope α, skewness β, and range γ of the standard parameterization of stable distribution [12,13]. These parameters are related to the asymptotics of $\rho(\lambda)$ at $\pm\infty$. Indeed, when

$$\rho(\lambda)\sim\frac{C_{\pm}}{\lambda^{1+\alpha}}\quad \lambda\rightarrow\pm\infty, \tag{5}$$

then $\beta=(C_+-C_-)/(C_++C_-)$ and $\gamma=\gamma_\alpha(C_++C_-)$, while α governs the powerlike asymptotics and γ_α is an α-dependent numerical coefficient.

$$b=\begin{cases}\gamma e^{i[(\alpha/2)-](1+\beta)\pi} & \text{for} \quad 1<\alpha<2\\ \gamma e^{i[\pi+(\alpha/2)(1+\beta)\pi]} & \text{for} \quad 0<\alpha<1\end{cases}.$$

In the marginal case $\alpha=1$, $R(z)$ reads

$$R(z)=a-i\gamma(1+\beta)-\frac{2\beta\gamma}{\pi}\ln\gamma z. \tag{6}$$

The branch cut structure of $R(z)$ is chosen in such a way that the upper complex half plane is mapped to itself. Recalling that $R=z-1/G$ in the large N limit, one finds that for the trivial case $R(z)=a$, the resolvent $G(z)=(z-a)^{-1}$, and the spectral distribution $\rho(\lambda)=\delta(\lambda-a)$. Otherwise, on the upper half plane, the resolvent fulfills an algebraic equation

$$bG^\alpha(z)-(z-a)G(z)+1=0, \tag{7}$$

or in the marginal case ($\alpha=1$),

$$[z-a+i\gamma(1+\beta)]G(z)+\frac{2\beta\gamma}{\pi}G(z)\ln\gamma G(z)-1=0. \tag{8}$$

On the lower half plane $G(\overline{z})=\overline{G}(z)$ [12]. The solution of the latter equation will not be discussed here, except for $\beta=0$ for which it simplifies to

$$G(z)=\frac{1}{z-(a-i\gamma)}. \tag{9}$$

Thus in this case the spectral density has the form of a Cauchy distribution (Lévy with $\alpha=1$),

$$\rho(\lambda)=\frac{1}{\pi}\frac{\gamma}{(\lambda-a)^2+\gamma^2}. \tag{10}$$

For $\alpha=1/4,1/3,1/2,2/3,3/4,4/3,3/2$, and 2 the algebraic equation (7) is exactly solvable. The ensuing distribution of eigenvalues obeys the scaling property $\rho_\gamma(\lambda)=\rho(\gamma^{1/\alpha}\lambda)/\gamma^{1/\alpha}$, with the asymptotic form $\rho(\lambda)\approx\sin(\pi\alpha/2)/\pi\lambda^{1+\alpha}$ (for $\beta=0$). Also $\rho_{\alpha,\beta}=\rho_{1/\alpha,\beta'}(1/x^\alpha)/x^{1+\alpha}$ for $1<\alpha<2$ and $\beta'=(\alpha-1)-(2-\alpha)\beta$ [13] and in analogy with the duality relations [14].

We now seek Hermitian ensembles with the measure (2), such that the corresponding $\rho(\lambda)$ coincides with the large N limit of the mean eigenvalue distribution. The resulting ensembles will then be automatically stable in the $N\rightarrow\infty$ limit. For the unitary ensemble, the standard procedure of diagonalizing $M\rightarrow U\Lambda U^\dagger$ and integrating out U gives rise to the standard joint probability distribution for the eigenvalues

$$\rho(\lambda_1,\ldots,\lambda_N)\prod_i d\lambda_i=\prod_i d\lambda_i e^{-NV(\lambda_i)}\prod_{i<j}(\lambda_i-\lambda_j)^2. \tag{11}$$

A similar distribution holds for symmetric and skew-symmetric ensembles. For large N, the corresponding partition function can be analyzed in terms of the Coulomb gas action with a continuous eigenvalue distribution $\rho(x)$ as originally suggested by Dyson [15]. Specifically,

$$\frac{S(\rho)}{N^2}=\int d\lambda\rho(\lambda)V(\lambda)-\int d\lambda d\lambda'\ln|\lambda-\lambda'|\rho(\lambda)\rho(\lambda'). \tag{12}$$

A functional and a standard differentiation yield,

$$V'(\lambda)=2\,\mathrm{P}\int d\lambda'\frac{\rho(\lambda')}{\lambda-\lambda'}, \tag{13}$$

where P denotes the principal value of the integral. The knowledge of $V(\lambda)$ plus the boundary conditions on the singular integral equation allow to find the spectral function [16]. Conversely, from the knowledge of the spectral function we can deduce the shape of the potential, hence the weight of the probability distribution for the matrix ensemble. This is the route for implementing the FRV results on random matrix ensembles as we now show.

Using the formula $1/(\lambda\pm i\varepsilon)=\mathrm{P}(1/\lambda)\mp i\pi\delta(\lambda)$, we get a formula for $V'(\lambda)$ in terms of the real part of the resolvent on the cut $V'(\lambda)=2\,\mathrm{Re}\,G(\lambda)$. We can now use the resolvents found in Sec. II to reconstruct the potential and hence, to explicitly define a random matrix realization of the free stable Lévy ensembles. Below, we reconstruct the pertinent measures for matrix analogs of the Cauchy ($\alpha=1,\beta=0$) and Lévy-Smirnov ($\alpha=1/2,\beta=\pm1$) distributions. In the cases $\alpha\neq 1/4,1/3,1/2,2/3,3/4,4/3,3/2$ such construction can be performed numerically. Some general results can be achieved without these specifics as we will show below.

In the case of the Cauchy distribution, elementary integration gives $V'(\lambda)=2\lambda/(\lambda^2+b^2)$, hence, the potential $V(\lambda)=\ln(\lambda^2+b^2)$. In the case of the Lévy-Smirnov distribution, the Green's function follows from

$$\frac{-i}{\sqrt{G}}+\frac{1}{G}=z, \tag{14}$$

which can be easily solved to yield

$$G(z)=\frac{2z-1-i\sqrt{4z-1}}{2z^2}. \tag{15}$$

Evaluating the real part gives $V'(\lambda)=2/\lambda-1/\lambda^2$, so that

$$V(M)=\exp\left[-N\,\mathrm{Tr}\left(\frac{1}{M}+\ln M^2\right)\right]. \tag{16}$$

The spectral function follows from Eqs. (3) and (21) and reads $\rho(\lambda)=1/(2\pi)\sqrt{4\lambda-1}/\lambda^2$.

Here, a word of caution is in order. In writing down formula (16) one has to restrict oneself to matrices that have non-negative eigenvalues. This assumption will be used in the sequel to construct the Coulomb gas representation of the Lévy-Smirnov ensemble. Instead of restricting the domain of matrices M to be explicitly positive definite one may represent all such matrices through $M=\sqrt{CC^\dagger}$, and integrate over the complex matrices C without any restrictions. If one follows this route an appropriate Jacobian for the change of variables $M=\sqrt{CC^\dagger}$ has to be included (Wishart measure [17]). The important point, however, is that the Coulomb gas representation remains unchanged.

In general, the asymptotic form of the potential for large eigenvalues reads

$$V(\lambda)=\ln\lambda^2-2\frac{1}{\alpha}\,\mathrm{Re}\,b\,\frac{1}{\lambda^{\alpha}}+\cdots, \tag{17}$$

In all Lévy cases the $\ln\lambda^2$ contribution in the potential is fixed, and is equivalent in the measure to $\det M$ with a fixed power $-2N$. As can be shown, a deviation from 2 leads to a finite support of eigenvalues. The coefficient of the second term in the potential can vanish in some notable cases such as, e.g., for the Lévy-Smirnov ensemble. In the next section we will analyze in greater detail the stable Lévy random matrix ensembles defined above.

Both the Cauchy and Lévy-Smirnov ensembles can be studied analytically at finite N using the orthogonal polynomial method. It turns out that the above ensembles have some unexpected features that do not appear in the classical case when $V(M)$ is a polynomial in M. For the Cauchy random matrix ensemble the orthogonal polynomials satisfy

$$\int d\lambda(\lambda^2+1)^{-2N}P_n(\lambda)P_m(\lambda)=\delta_{nm}. \tag{18}$$

We see that in contrast to the classical case only a *finite* number of orthogonal polynomials exist. These are explicitly given by Jacobi polynomials analytically continued to complex parameters. (After completing the paper, we noticed that a similar construction was recently used in [18].) Indeed,

$$P_n(x)=\left(\frac{(1+n-2N)_n}{2^{2n}n!}\sqrt{\pi}\,\frac{\Gamma\left(N-n-\frac{1}{2}\right)}{\Gamma(N-n)}\right)^{-1/2}\times i^n J_n^{-N,-N}(ix). \tag{19}$$

A second surprise comes from the fact that the eigenvalue distribution is exactly equal to $\rho(\lambda)=1/\pi(\lambda^2+1)$, and does *not* depend on N. There are no finite N corrections whatsoever to the spectral distribution. In particular the classical short distance oscillations in the spectral density characteristic for random matrix models are absent. The Cauchy ensemble has, however, nontrivial N-dependent two-point correlation functions.

The Lévy-Smirnov ensemble can be analyzed starting from the distribution (11) with the measure (16)

$$\prod_i d\lambda_i\left(\frac{e^{-N/\lambda_i}}{\lambda^{2N}}\right)\prod_{i<j}(\lambda_i-\lambda_j)^2. \tag{20}$$

In the above formula the λ_i's are assumed to be *non-negative*. A change of variables $\lambda_i=1/x_i$ leads to

$$\prod_i dx_i(e^{-Nx_i})\prod_{i<j}(x_i-x_j)^2. \tag{21}$$

This is readily analyzed in terms of Laguerre polynomials similar to chiral Gaussian unitary ensembles (chGUE) [19]. Indeed the appropriate polynomials are

$$P_n(x)=\sqrt{N}L_n^0(Nx). \tag{22}$$

The eigenvalue density can be written, using the Christoffel-Darboux identity, as

$$\rho(x)=Ne^{-Nx}[L^0_{N-1}(Nx)L^1_{N-1}(Nx)-L^0_N(Nx)L^1_{N-2}(Nx)]. \tag{23}$$

In particular we note that there exists a well defined microscopic limit that corresponds to expressing the eigenvalue density in terms of $x=s/N^2$ (i.e., on the scale of the eigenvalue spacing). The relevant eigenvalue density $(1/N)\rho(s/N^2)$ is then given by

$$\rho(s)=J_0^2(2\sqrt{s})+J_1^2(2\sqrt{s}). \tag{24}$$

Going back to the original variables, the microscopic region corresponds to the region of *large* eigenvalues $\lambda=N^2\Lambda$ in the powerlike tail. For these *large* eigenvalues we therefore observe chGUE-like oscillations

$$\rho(\Lambda)=\frac{1}{\Lambda^2}\left\{J_0^2\left(\frac{2}{\sqrt{\Lambda}}\right)+J_1^2\left(\frac{2}{\sqrt{\Lambda}}\right)\right\}. \tag{25}$$

Moreover, we expect these oscillations to be *universal* in the following sense. A generic modification of the LS potential of the form

$$V(\lambda)=\log\lambda^2+\frac{1}{\lambda}+\frac{g_2}{\lambda^2}+\frac{g_3}{\lambda^3}+\cdots, \tag{26}$$

will not change the oscillation pattern (25). This follows from the results in [21] after the change of variables $\lambda\to x=1/\lambda$. The coefficient in front of the logarithm cannot be changed, for otherwise the eigenvalue support becomes finite and the powerlike tails disappear altogether. The coefficients of the $1/\lambda$ term (and higher g_i's) only affect the length scale of the universal oscillations.

In the general case when the potential is of the form (17), and the asymptotic behavior of $\rho(\lambda)$ is similar to $1/\lambda^{1+\alpha}$, the mapping $\lambda\to x=1/\lambda$ gives an effective potential $V(x)\sim -2\,\mathrm{Re}\,bx^{\alpha}/\alpha$, and an eigenvalue distribution $\rho(x)\sim x^{\alpha-1}$. General arguments [22] show that the resulting eigenvalue spacing $1/N^{1/\alpha}$ yields a microscopic distribution in the limit of $N\to\infty$ with $s=xN^{1/\alpha}$ fixed. The pertinent orthogonal polynomials should satisfy

$$\int dx e^{-NV(x)}P_n(x)P_m(x)=\delta_{nm}, \tag{27}$$

with $V(x)\sim -2\,\mathrm{Re}\,bx^{\alpha}/\alpha+\cdots$. Here, the next-to-leading terms have to be included, as the vanishing or singular behavior of $\rho(x)$ at $x\sim 0$ requires fine tuning of the subleading coefficients (for the case without fine tuning, and hence without a multicritical regime see the interesting paper of [20]). All this is very reminiscent of multicritical microscopic scaling and universality [22–24] in the classical random matrix case but here the ‘‘multicritical classes’’ are labeled by a *real* parameter α and not by an *integer*. A thorough investigation of this regime seems to be very interesting. This behavior may be relevant for studying the critical behavior of QCD at the chiral restoration point [22,23], in light of the fact that the lattice data suggest non-mean-field critical exponents.

Let us comment briefly on the comparison between the matrix ensemble discussed in [8] and ours. The ensemble [8] by construction is not rotationally invariant, therefore, is not stable under the matrix convolution law. It is also not amenable to several mathematical methods developed in standard random matrix theory. As rotationally noninvariant, it exhibits correlations between the eigenvectors (contrary to GUE, GOE, GSpE, as well as the case considered here), which nevertheless, might be interesting phenomenologically. The only study of both ensembles known to us deals with some bulk properties of financial data and the results are, overall, similar [25]. The comparison of the higher-point correlation functions in both models requires much more analytical and numerical work, which goes beyond the scope of the present paper.

Finally, we will speculate on the potential relevance of the present study to the statistical analysis of evolving networks (for recent reviews, see [26,27]). Indeed, it was realized recently that most of the large artificial networks (e.g., Internet) as well as biophysical and socioeconomic networks (so-called scale-free networks) display unusual spectral properties. In the case of the classical networks (random graph theory of Erdös and Renyi [28]) the spectral properties are given by the semicircle law of GOE, whereas in the case of scale-free networks the spectral density of the adjacency matrices is a power law, with almost universal exponents belonging to the Lévy stability window. Recent and independent studies of large data sets using random network covariances [29] and financial covariances [25] show close relation to the random Levy matrix theory discussed here, with even similar power-law exponents. Therefore, it is tempting to conjecture that the random Levy matrix theory is for scale-free networks, whereas Gaussian random matrix theory is for classical random graphs.

We have explicitly constructed matrix realizations of free random variables, with potential applications to a number of stochastic phenomena. This opens several venues for applying FRV calculus to Lévy processes, including convolution, multiplication and addition of deterministic matrixlike entries, and other generalizations. Using the Coulomb gas analogy, we have shown that the exact matrix measure in the case of powerlike spectra is nonlocal (involves determinants). The construction exhibits several nontrivial features, among which the most interesting ones are a universal behavior in the tails of the distributions and an unusual large N scaling. The expected microscopic eigenvalue distribution defines a universal regime and represents a generalization of the multicritical scaling discussed in [22–24]. We also pointed out the possible relevance of our results for the rapidly growing field of scale-free networks. Several of these issues, as well as practical applications of our results are discussed in subsequent work [25].

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